TABLE II. $14 \rightarrow 7+7$ decays.

$I^+ \rightarrow pK^0(a)$	$nK^+(a)$					
$N_*^{++} \rightarrow p\pi^+(3b)$	$\Sigma^+K^+(3c)$					
$N_*^+ \rightarrow p\pi^0(2b)$	$n\pi^+(b)$	$\Sigma^+K^0(c)$	$\Sigma^0K^+(2c)$			
$N_*^0 \rightarrow n\pi^0(2b)$	$p\pi^-(b)$	$\Sigma^-K^+(c)$	$\Sigma^0K^0(2c)$			
$N_*^- \rightarrow n\pi^-(3b)$		$\Sigma^-K^0(3c)$				
$Y_*^+ \rightarrow pK^0(2g)$	$\Sigma^+\pi^0(h)$	$\Sigma^0\pi^+(h)$	$\Xi^0K^+(2j)$			
$Y_{1*}^0 \rightarrow pK^-(g)$	$n\bar{K}^0(g)$	$\Sigma^+\pi^-(h)$	$\Sigma^-\pi^+(h)$	$\Xi^0K^0(j)$	$\Xi^-K^+(j)$	
$Y_{0*}^0 \rightarrow pK^-(k)$	$n\bar{K}^0(k)$	$\Sigma^+\pi^-(l)$	$\Sigma^0\pi^0(l)$	$\Sigma^-\pi^+(l)$	$\Xi^0K^0(m)$	$\Xi^-K^+(m)$
$Y_*^- \rightarrow nK^-(2g)$	$\Sigma^-\pi^0(h)$	$\Sigma^0\pi^-(h)$	$\Xi^-K^0(2j)$			
$\Xi_*^+ \rightarrow \Xi^0\pi^+(3e)$	$\Sigma^+\bar{K}^0(3d)$					
$\Xi_*^0 \rightarrow \Xi^0\pi^0(2e)$	$\Xi^-\pi^+(e)$	$\Sigma^+K^-(d)$	$\Sigma^0\bar{K}^0(2d)$			
$\Xi_*^- \rightarrow \Xi^-\pi^0(2e)$	$\Xi^0\pi^-(e)$	$\Sigma^-\bar{K}^0(d)$	$\Sigma^0K^-(2d)$			
$\Xi_*^{--} \rightarrow \Xi^-\pi^-(3e)$	$\Sigma^-K^-(3d)$					
$\Omega^- \rightarrow \Xi^-\bar{K}^0(f)$	$\Xi^0K^-(f)$					

get six modified Shmushkevich equations:

$$
\begin{aligned}
2(a+f)=3(b+c+d+e)&=4(g+h+j)\\
&=4(k+m)+6l,\\
2(a+g+h+j)&=6(b+c),\\
a+6b+3g+k+3j+m+6e+f&=8c+8d+4h+2l,\\
a+6c+3j+m+3g+k+6d+f&=8b+8e+4h+2l.
\end{aligned}
\quad (9)
$$

From these equations we can deduce

$$3(b+e)=2(g+h+j).$$

So that we have the inequality

$$b+e>\tfrac{2}{3}g, \quad (10)$$

which may be experimentally tested.

PHYSICAL REVIEW VOLUME 133, NUMBER 2B 27 JANUARY 1964

A Field Theory of Weak Interactions.* II

G. FEINBERG†
Physics Department, Columbia University, New York, New York

AND

A. PAIS
The Rockefeller Institute, New York, New York

(Received 22 August 1963)

The study of the Bethe-Salpeter equation for lepton-lepton interaction mediated by charged vector mesons is continued. Starting from the exact integral equation, an improved approximation procedure is developed. This reproduces the low-energy results for "allowed" processes given in a previous paper. Beyond that, it is now found that to leading order the BS equation gives the value $3g^2/4\pi^2$ for the zero-energy ratio between "forbidden" and "allowed" amplitudes, where g is the bare meson-lepton coupling constant. Some information on the momentum dependence of the forbidden amplitude is also obtained. The mathematical methods developed in an earlier paper are then applied to the corresponding Bethe-Salpeter equation of the Fermi field theory. It is shown that the calculated amplitudes for both allowed and forbidden processes are equal to zero. This illustrates the fact that if higher order effects are taken seriously, there is no reason to consider the Fermi field theory as the limiting case of a vector meson theory with a boson mass which tends to infinity.

1. INTRODUCTION

IN the first paper in this series,[1] we have shown that in the vector meson theory of weak interactions (W theory), graphs involving more than one virtual vector meson may give sizable contributions to the matrix element for processes like μ decay. This is true in spite of the smallness of the meson-lepton coupling constant g, and occurs because of the divergences which make the perturbation expansion meaningless. In I we

* Work supported in part by the U. S. Atomic Energy Commission.
† Alfred P. Sloan Foundation Fellow.
[1] G. Feinberg and A. Pais, Phys. Rev. 132, 2724 (1963), referred to in this paper as I. We denote Eq. (4.19) of that paper by Eq. (I4.19) in this work. For the terminology "leptonic" and "semileptonic" see I, Ref. 2.

also gave a preliminary discussion of semileptonic reactions. In the present paper we confine ourselves to leptonic phenomena only.

In I, we considered the set of uncrossed ladder graphs[2] for lepton-lepton scattering, and derived the Bethe-Salpeter (BS) equation for the scattering amplitude described by these graphs. This equation was then solved by an iteration scheme in which the first approximation includes the apparently most singular term in the kernel of the BS equation. We called this first approximation to the exact BS equation the "approximate integral equation," see Eq. (I4.22). The remaining terms in the equation were then treated by successive approximations and were found to be small. In this way, we succeeded in showing that for some physical processes (the "allowed" ones) the contribution of the higher order graphs is finite and comparable to the one meson exchange, whereas for other processes (the "forbidden" ones), the contribution of higher order graphs is finite but small compared with g^2.

In this second paper, we wish to present a somewhat different treatment of the integral equation for the uncrossed ladder graphs in the W theory.[3] There are several reasons which impel us to do this.

(1) According to the method developed in I, the leading correction due to higher order effects is obtained by summing the "most divergent" terms in the perturbation expansion of the graphs considered. It is not evident that this prescription is equivalent to solving the approximate integral equation. One way of judging this would be to know whether or not the rather complicated iteration procedure outlined in I actually converges, but this we were unable to ascertain. In this paper we show that the answer obtained in I is nevertheless correct to $O(g^2)$.

This conclusion has meanwhile been reached independently by Pwu and Wu.[4] The present work goes beyond the results of these authors in that we also obtain, at zero energy, the magnitude of the correction term. This term is of some importance, as it determines to *leading* order the magnitude of the "forbidden" processes. In the Pwu-Wu approximation the only result obtained for the forbidden amplitude is: zero to $O(g^2)$.

(2) In I we found,[5] up to the accuracy of the first iteration, that the forbidden amplitude is at most of order $g^4 \ln g$. To get the coefficient of the leading term in the forbidden process, it would at least be necessary to calculate the first iteration precisely. This is rather cumbersome and was not carried out in detail there. In this paper, we present a considerably simpler method for obtaining the zero-energy forbidden amplitude. We shall see that the $g^4 \ln g$ terms which previously appeared in various stages of the calculation are spurious. The actual order of the forbidden amplitude will here be found to be g^4.

In order to get around the difficulties mentioned above it is evidently indicated to return to the exact BS equation which we gave in Eq. (IA.8) and which we restate in the next section. As in I, we set the lepton masses equal to zero in the kernel of this equation. By the general power counting argument,[6] we know that any g^2 corrections to the lowest order amplitude can be obtained by setting all external momenta equal to zero. To make use of this, we set the initial momenta p_1,p_2 equal to zero. The exact integral equation then simplifies considerably since it now depends on a single four-vector momentum p. This "zero-momentum" equation, suitably regularized, is studied in some detail in Sec. 3. We find that the solution can be split into two parts, a "high-energy part" and a "low-energy" part. The high-energy part contains the divergences of the perturbation solution, and is actually a constant, independent of momentum for any value of the regulator mass. In the limit of infinite regulator mass, the constant goes to zero, and the high-energy part vanishes. In the discussion of the high energies we meet again (as is to be expected) with the "peratization procedures" which we outlined in I, though now slightly different techniques are used. However, the point remains the same, to wit, that the higher order effects are not negligible, on the one hand, but not explosive either, on the other, due to the self-damping at high energies.

The remaining low-energy part satisfies an integral equation which can be iterated, giving finite results. That is, the solution can be expanded in powers of g^2, each term having a finite coefficient. The leading (g^2) term agrees at $p=0$ with the solution found in I, i.e., the lowest order amplitude for allowed processes is reduced by a factor 3/4. Since the corrections at zero momenta can be expanded in powers of g^2, we get no $g^4 \ln g$ in the expression for the "forbidden" amplitudes. Instead we find the ratio of forbidden to allowed amplitude at zero momentum to be $(3g^2/4\pi)+O(g^4)$. Some further remarks on the amplitudes at finite momenta are made at the end of Sec. 2(c) and in Appendix B.

In Sec. 3 we apply our techniques to a field theory with local four-fermion interactions. In lowest order, this theory can of course be considered as the limit of the W theory for infinite boson mass, in the following sense. Let

$$m \longrightarrow \infty\,,\quad g \longrightarrow \infty\,,$$

but keep the ratio of these two quantities constant in such a way that

$$g^2/m^2 = G/\sqrt{2}\,,$$

[2] These are defined as the graphs in which the arrows on the fermion lines are parallel and the meson lines are uncrossed, see I, Sec. (VII.b).

[3] We use again the Lagrangian Eq. (I2.1).

[4] Y. Pwu and T. T. Wu, University of Pennsylvania (unpublished).

[5] See the Summary in I, Sec. VI(e). Computational details are found in I, Appendix F.

[6] See I, Secs. III and VII(a).

where G is the Fermi constant. However, this simple relation between the W and the Fermi theory is by no means maintained if higher order effects are included. This is illustrated in Sec. 4, where we apply the identical methods of Sec. 3 to the equivalent problem in the Fermi theory. That is to say, we deal with the set of graphs (bubble graphs) which is obtained from the uncrossed ladder graphs of the W theory by pinching the W rungs to a point. Upon summing these bubble graphs in this way, we find that the amplitudes for both allowed and forbidden processes go to zero as the regulator mass goes to infinity.[7] This lack of correspondence with the W theory is evidently due to the fact that in computing higher order graphs, one integrates over momenta which can be large compared to the mass m.

We are in no position to say that the vanishing of all amplitudes proves that a Fermi field theory is meaningless, as we do not know whether the inclusion of larger sets of graphs may change this result. However, we believe that the present comparison between the W theory and the Fermi theory is instructive. It is well known that both of these theories are unrenormalizable. In the light of our present results it seems to us that this statement is as enlightening as saying that a purple horse is as probable as a green cow.

As in I, we use regulator procedures in order to have finite integrals over momentum space throughout. Only after all such integrations have been performed do we let the regulator mass tend to infinity. However, unlike the method used in I, we do not regulate the W propagator at all, but rather the lepton propagators. We were led to this by the study of the Fermi field theory. It then turned out to be technically expedient to use the same method for the W theory. In particular this makes the comparison of the two theories quite transparent at every stage. Presumably the particular choice of regularization method does not affect the final answer, however.

2. MOMENTUM DEPENDENCE OF THE AMPLITUDES

We consider here the dependence on external momenta of the various amplitudes. We use the same notations as in I, see especially I, Sec. IV and Appendix A. The allowed and forbidden amplitudes will generally be of the form

$$M_{\mu\nu}\bar{u}\gamma_\mu(1+\gamma_5)u\ \bar{u}\gamma_\nu(1+\gamma_5)u\,, \tag{2.1}$$

where each one of the various spinors u has in each specific case its appropriate particle label. We noted earlier that the lepton mass is neglected in the kernel of the BS equation. It then follows from γ_5 invariance that there can be no scalar, tensor terms induced by the higher order weak effects.

The tensor $M_{\mu\nu}$ can be written as a sum of terms

$$M_{\mu\nu}=\alpha\delta_{\mu\nu}+\beta q_\mu q_\nu+m_{\mu\nu}\,, \tag{2.2}$$

where we have isolated the first two terms because they are the only ones which get contributions from the exchange of one vector meson. The form factors α, β are scalar functions of q^2 and P^2 (q=momentum transfer, P=total energy four vector). The remaining terms $m_{\mu\nu}$ can be constructed from the available independent 4-vectors, which we can take as q_μ, $p_{1\mu}$, and $p_{2\mu}$. For later purposes we note that $m_{\mu\nu}$ has the following property: If we set $p_1=p_2=0$, and $p_1'+p_2'=0$, but not p_1', p_2' separately zero, then $m_{\mu\nu}$ vanishes, whereas the terms in α and β do not vanish. These conditions can only be satisfied off the mass shell, as is easily seen in the c.m. system of the incident particles.

In second-order perturbation theory, we have for the allowed process

$$\alpha_2=\alpha(q^2)=-ig^2/(q^2+m^2)\,, \tag{2.3}$$

$$\beta_2=\beta(q^2)=-ig^2/m^2(q^2+m^2)\,, \tag{2.4}$$

$$m_{\mu\nu,2}=0\,. \tag{2.5}$$

According to the general power counting argument,[6] the only quantity which gets a g^2 correction from multimeson exchange is α, and furthermore this correction is independent of momentum. That is (always for the allowed process) to order g^2,

$$\alpha=\alpha_2+i\eta g^2/m^2\,, \tag{2.6}$$

$$\beta=\beta_2\,, \tag{2.7}$$

$$m_{\mu\nu}=0\,, \tag{2.8}$$

where η is a constant.

If we are prepared to accept this conclusion based on power counting, rather than to demonstrate it explicitly as we did through our iteration procedure in I, it is evidently sufficient to calculate β, and α with all momenta set equal to zero, since

$$(ig^2/m^2)\eta=\alpha(p_{\text{ext}}=0)+(ig^2/m^2)\,. \tag{2.9}$$

It should again be emphasized[6] that the Eqs. (2.6)–(2.9) have a much more general validity than for ladder graphs only. It was only for the latter subset of graphs that a value for η could be given in I, namely,

$$\eta=\tfrac{1}{4}\,. \tag{2.10}$$

The general quantity η remains unknown so far,[8] however, and it may be noted that η is *not* additively composed of contributions from ladder graphs and from "other" graphs. As was already noted in I, it is highly desirable to extend to larger classes than only the

[7] This result is true for a fixed value of the bare Fermi constant G. We shall not enter here in discussions of limiting process which are of the type: $G\to\infty$, a cutoff $\to\infty$, in some prescribed ratio. For such arguments we refer to A. Abrikosov, A. Galanin, L. Gorkov, L. Landau, I. Smorodinsky, and K. Ter-Martirosyan, Phys. Rev. **111**, 321 (1958).

[8] However, in I Sec. VI(d) we gave reasons for a conjecture that Eq. (1.10) may well have a much wider validity.

uncrossed ladder graphs the general method of resumming graphs by ordering the contributions with respect to their degree of singularity. In this paper we have nothing further to say about this subject.

To return to the ladder graphs, we set $p_1=p_2=0$ from the start. Because of this restriction, we cannot here use the BS equation to say anything definite about $m_{\mu\nu}$, which vanishes at zero momentum, or about momentum dependent corrections to α, β except that they are of higher order in g^2. It is, however, possible to compute the leading corrections to α, β, and $m_{\mu\nu}$ at low momenta by another method, providing that we believe that the theory really is well defined. We return to this in Appendix B.

The information about allowed processes is contained in Eqs. (2.6)–(2.10). For the forbidden processes we shall find the following zero energy values for α and β.

$$\alpha=\frac{9ig^4}{16\pi^2m^2},\quad \beta=\frac{ig^4}{8\pi^2m^4}. \tag{2.11}$$

The results for the Fermi case, to be discussed in Sec. 4, are expressed by

$$\alpha=\beta=m_{\mu\nu}=0 \tag{2.12}$$

for all energies.

3. BS EQUATION IN THE W THEORY

(a) Reduction of the Integral Equation

As in I we denote the amplitudes for the allowed and forbidden processes by M_{odd} and M_{even}, respectively. We introduce again the linear combinations

$$M_\pm=M_{\text{odd}}\pm M_{\text{even}} \tag{3.1}$$

and put

$$M_\pm=M_{\mu\nu}{}^\pm\gamma_\mu{}^{(1)}(1+\gamma_5{}^{(1)})\gamma_\nu{}^{(2)}(1+\gamma_5{}^{(2)}). \tag{3.2}$$

The superscripts (1), (2) distinguish the Dirac matrices on the two fermion lines. We next restate the integral equation which $M_{\mu\nu}{}^\pm$ was found to satisfy in I, see Eq. (IA.8).

$$\begin{aligned}M_{\mu\nu}{}^\pm(p_1',p_2',p_1,p_2)&=-ig^2\left[\delta_{\mu\nu}+\frac{(p_1'-p_1)_\mu(p_1'-p_1)_\nu}{m^2}\right]\frac{1}{(p_1'-p_1)^2+m^2}\\&\mp\frac{4ig^2}{(2\pi)^4}\xi_{\alpha\beta\rho\mu}\xi_{\sigma\tau\lambda\nu}\int\frac{p_{1\beta}''p_{2\tau}''}{(p_1'')_R{}^2(p_2'')_R{}^2}\\&\times\left[\delta_{\alpha\sigma}-\frac{(p_1''-p_1')_\alpha(p_2''-p_2')_\sigma}{m^2}\right]\\&\times\frac{1}{(p_1''-p_1')^2+m^2}M_{\rho\lambda}{}^\pm(p_1'',p_2'',p_1,p_2)d^4p_1''.\end{aligned} \tag{3.3}$$

The meaning of the symbols are as follows. p_1, p_2 are the two incoming four-momenta, p_1', p_2' the outgoing ones, while

$$p_2''=p_1+p_2-p_1''. \tag{3.4}$$

The ξ symbols are defined by

$$\xi_{\alpha\beta\rho\mu}=\delta_{\alpha\beta}\delta_{\rho\mu}-\delta_{\alpha\rho}\delta_{\beta\mu}+\delta_{\alpha\mu}\delta_{\beta\rho}+\epsilon_{\alpha\beta\rho\mu}. \tag{3.5}$$

The following identities are easily derived.[9]

$$\xi_{\alpha\beta\rho\mu}A_{\rho\mu}=\delta_{\alpha\beta}A_{\rho\rho}\quad\text{if}\quad A_{\rho\mu}=A_{\mu\rho}, \tag{3.6}$$

$$\xi_{\alpha\beta\rho\mu}\xi_{\sigma\tau\rho\mu}=4\xi_{\alpha\beta\tau\sigma}, \tag{3.7}$$

$$\xi_{\alpha\beta\rho\mu}\xi_{\alpha\tau\rho\sigma}=4\delta_{\beta\tau}\delta_{\mu\sigma}, \tag{3.8}$$

$$\xi_{\alpha\beta\rho\mu}\xi_{\sigma\tau\rho\nu}X_\alpha X_\sigma Y_\beta Y_\tau=X_\alpha{}^2Y_\beta{}^2\delta_{\mu\nu}. \tag{3.9}$$

In one respect, Eq. (3.3) differs from the integral equation given in Eq. (IA.8). In the latter we had regulated the boson propagators $[(p_1'-p_1)^2+m^2]^{-1}$ in the inhomogeneous term and $[(p_1''-p_1')^2+m^2]^{-1}$ in the kernel. We have not done so here but have regulated the fermion propagators instead, as is indicated by the notation $1/(p_1'')_R{}^2$ in Eq. (3.3). Here we define

$$\frac{1}{(p^2)_R}=\frac{1}{p^2}-\frac{1}{p^2+M^2}. \tag{3.10}$$

More precisely this means that the fermion propagator has been treated as follows. We neglect a possible lepton mass and put

$$\left(\frac{1}{p\!\!\!/}\right)_R=-p\!\!\!/\left(\frac{1}{p^2}-\frac{1}{p^2+M^2}\right),\quad p\!\!\!/=-i\gamma_\lambda p_\lambda. \tag{3.11}$$

This procedure maintains the γ_5 invariance throughout.

In this paper we determine the zero-energy behavior of $M_{\mu\nu}{}^\pm$, and therefore put $p_1=p_2=0$ from the start. Then

$$\begin{aligned}p_1'&=-p_2'\equiv p,\\p_1''&=-p_2''\equiv p',\end{aligned} \tag{3.12}$$

and

$$M_{\mu\nu}{}^\pm\to M_{\mu\nu}(p,M) \tag{3.13}$$

depends on a single four-momentum. As we have stated, on the mass shell $p=0$, so that to get the physical amplitude we must set $p=0$. In particular $M_{\mu\nu}(p,M)$ does not represent a physical quantity in the scattering problem. It would nonetheless be of interest to compute it as a function of p, as it may occur in iteration methods for calculating other matrix elements of interest such as the propagator, vertex etc. We return to this question elsewhere.

It is helpful for what follows to write the quantities on the right-hand side of Eq. (3.13) explicitly as a function of a four-momentum as well as of the regulator mass M. On covariance grounds

$$M_{\mu\nu}{}^\pm(p,M)=\alpha^\pm(p,M)\delta_{\mu\nu}+\beta^\pm(p,M)p_\mu p_\nu. \tag{3.14}$$

[9] Equation (3.9) was independently derived by Dr. T. T. Wu.

Insert Eqs. (3.10) and (3.12)–(3.14) into Eq. (3.3) and use the identities Eqs. (3.6)–(3.9). One finds

$$\alpha^{\pm}(p,M)\delta_{\mu\nu}+\beta^{\pm}(p,M)p_\mu p_\nu$$
$$=\frac{-ig^2(\delta_{\mu\nu}+m^{-2}p_\mu p_\nu)}{p^2+m^2}\pm\frac{4ig^2M^4}{(2\pi)^4}$$
$$\times\int\frac{d^4p'}{p'^2(p'^2+M^2)^2[(p'-p)^2+m^2]}$$
$$\times\{\delta_{\mu\nu}[4\alpha^{\pm}(p',M)+p'^2\beta^{\pm}(p',M)]$$
$$+(1/m^2)[(p'-p)^2\alpha^{\pm}(p',M)\delta_{\mu\nu}$$
$$+(p'-p)_\mu(p'-p)_\nu p'^2\beta^{\pm}(p',M)]\}. \quad (3.15)$$

Take the trace of Eq. (3.15) and define

$$T^{\pm}(p,M)=4\alpha^{\pm}(p,M)+p^2\beta^{\pm}(p,M). \quad (3.16)$$

$T^{\pm}(p,M)$ is found to satisfy

$$T^{\pm}(p,M)=\frac{-ig^2(4+m^{-2}p^2)}{p^2+m^2}$$
$$\pm\frac{4ig^2M^4}{(2\pi)^4}\int\frac{d^4p'}{p'^2(p'^2+M^2)^2[(p'-p)^2+m^2]}$$
$$\times\left[4+\frac{(p'-p)^2}{m^2}\right]T^{\pm}(p',M). \quad (3.17)$$

Use Eqs. (3.16), (3.17) to substitute for $\alpha^{\pm}$ back in Eq. (3.15). This yields

$$A_{\mu\nu}(p)\beta^{\pm}(p,M)=\frac{-ig^2}{m^2}A_{\mu\nu}(p)\frac{1}{p^2+m^2}\pm\frac{4ig^2M^4}{(2\pi)^4m^2}$$
$$\times\int\frac{d^4p'}{(p'^2+M^2)^2[(p'-p)^2+m^2]}$$
$$\times A_{\mu\nu}(p-p')\beta^{\pm}(p',M), \quad (3.18)$$

where

$$A_{\mu\nu}(p)=p_\mu p_\nu-\tfrac{1}{4}\delta_{\mu\nu}p^2. \quad (3.19)$$

Multiply Eq. (3.18) by

$$A_{\rho\mu}^{-1}=-(4/p^2)\left(\delta_{\rho\mu}-\frac{4}{3}\frac{p_\rho p_\mu}{p^2}\right),$$

and take the trace. The result is

$$\beta^{\pm}(p,M)=\frac{-ig^2}{m^2(p^2+m^2)}\pm\frac{16ig^2M^4}{3(2\pi)^4m^2}\frac{A_{\mu\nu}(p)}{p^4}$$
$$\times\int\frac{(p-p')_\mu(p-p')_\nu\beta^{\pm}(p')d^4p'}{(p'^2+M^2)^2[(p-p')^2+m^2]}. \quad (3.20)$$

Thus we have now found that the exact BS equation reduces to the uncoupled integral equations (3.17), (3.20) in the zero energy limit.

Before we proceed to discuss the solutions to these equations, let us first define precisely the quantities we are ultimately interested in. This has yet to be done, because we have to state in which order the limiting processes $p\to 0$, $M\to\infty$ have to be performed. We do this in the order just indicated, that is, we define

$$T^{\pm}=\lim_{M\to\infty}\lim_{p\to 0}T^{\pm}(p,M) \quad (3.21)$$

as the physical value of the trace at zero energy.

This can be made plausible as follows: The physical matrix element $M^{\pm}$ of Eq. (3.1) represents the sum of contributions of graphs of various orders. As was explained in I, Sec. III, these contributions are, always for zero energy, functions of g, m, and M. We have to find this sum for finite M. In the present paper this is done by first introducing quantities like $T^{\pm}(p,M)$ which do not only depend on M but also on the unphysical (off the mass shell) parameter p, and then letting $p\to 0$. After that is done we must finally let $M\to\infty$. This is the content of Eq. (3.21).

We show in Sec. 3(c) that

$$\lim_{M\to\infty}\lim_{p\to 0}p^2\beta^{\pm}(p,M)=0. \quad (3.22)$$

From Eqs. (3.16, 3.21, 3.22)

$$\alpha^{\pm}\equiv\lim_{M\to\infty}\lim_{p\to 0}\alpha^{\pm}(p,M)=\tfrac{1}{4}T^{\pm}, \quad (3.23)$$

so that at zero energy, and neglecting the lepton masses

$$M^{\pm}=\alpha^{\pm}\gamma_\mu^{(1)}(1+\gamma_5^{(1)})\gamma_\mu^{(2)}(1+\gamma_5^{(2)}). \quad (3.24)$$

(b) Iteration Procedure for the Trace Equation

Put

$$T^{\pm}(p,M)=T_1^{\pm}(p,M)+T_2^{\pm}(p,M),$$

where

$$T_1^{\pm}(p,M)=\frac{-ig^2}{m^2}\pm\frac{4ig^2M^4}{(2\pi)^4m^2}$$
$$\times\int\frac{d^4p'[T_1^{\pm}(p',M)+T_2^{\pm}(p',M)]}{p'^2(p'^2+M^2)^2} \quad (3.26)$$

$$T_2^{\pm}(p,M)=\frac{-3ig^2}{p^2+m^2}\pm\frac{12ig^2M^4}{(2\pi)^4}$$
$$\times\int\frac{d^4p'T_1^{\pm}(p',M)}{p'^2(p'^2+M^2)^2[(p-p')^2+m^2]}\pm\frac{12ig^2M^4}{(2\pi)^4}$$
$$\times\int\frac{d^4p'T_2^{\pm}(p',M)}{p'^2(p'^2+M^2)^2[(p'-p)^2+m^2]}. \quad (3.27)$$

Equations (3.25)–(3.27) are equivalent to Eq. (3.17). Let us consider some general features of Eqs. (3.26)–(3.27).

Equation (3.26) is evidently solved by

$$T_1^{\pm}(p,M)=T_1^{\pm}(M)\,,\quad \text{independent of } p. \tag{3.28}$$

This means that $T_1^{\pm}$ will persist at high virtual frequencies, *unless it is equal to zero at all frequencies*. The quantity $T_1^{\pm}(M)$ is what we called the "high-energy part" in the Introduction. We show that $T_1^{\pm}(M)$ indeed $\rightarrow 0$ as we let $M\rightarrow\infty$ after having performed all integrations.

Generally, it follows from Eq. (3.28) that we may look upon the first two terms on the right-hand side of Eq. (3.27) as "the inhomogeneity" of the integral equation for $T_2^{\pm}(p,M)$. Because $T_2^{\pm}(p,M)\rightarrow 0$ as $p\rightarrow\infty$ we referred to it as the "low-energy part" in the Introduction.

We now proceed as follows. First we solve Eq. (3.27) by iteration, considering its inhomogeneity as the leading solution, then taking this inhomogeneity substituted in the last term of Eq. (3.27) as the next correction, etc. In other words we do Born approximation *for finite M*. The solution thus found still depends on the unknown quantity $T_1^{\pm}(M)$ of Eq. (3.28) which then is of course determined by Eq. (3.26). Thus after we have solved for $T_2^{\pm}(p,M)$ to any desired order in g, we can solve *rigorously* Eq. (3.26) for $T_1^{\pm}(M)$. We pursue this method for a few steps.

First of all Eq. (3.27) gives the following result to leading order for $T_2^{\pm}(0,M)$:

$$T_2^{\pm}(0,M)=-\frac{3ig^2}{m^2}\mp\frac{3g^2T_1^{\pm}(M)}{2\pi^2}\ln\frac{M}{m}. \tag{3.29}$$

Note that some useful Feynman integrals used here and in the following are collected in Appendix A.

In order to know the limit value for $T_2^{\pm}(0,M)$ as $M\rightarrow\infty$ we must of course find out how $T_1^{\pm}(M)$ behaves with M. This we have done to the accuracy where one substitutes the g^2 terms of $T_2^{\pm}(p,M)$ into Eq. (3.26). This yields

$$T_1^{\pm}(M)=\left(-\frac{ig^2}{m^2}\pm\frac{3ig^4}{2\pi^2m^2}\ln\frac{M}{m}\right)\Bigg/\left[1\pm\frac{g^2}{4\pi^2}\left(\frac{M}{m}\right)^2+\frac{48g^4M^8}{m^2}\phi(M)\right], \tag{3.30}$$

where

$$\phi(M)=\frac{1}{(2\pi)^8}\int\frac{d^4p}{p^2(p^2+M^2)^2}\times\int\frac{d^4q}{q^2(q^2+M^2)^2[(q-p)^2+m^2]}. \tag{3.31}$$

We estimate $\phi(M)\sim 1/M^6$. Hence

$$T_1^{\pm}(M)\rightarrow 0\quad\text{as}\quad 1/M^2\,, \tag{3.32}$$

so that to leading order, see Eqs. (3.14) and (3.29),

$$M_{\mu\nu}^{\pm}=-(3ig^2/4m^2)\delta_{\mu\nu}. \tag{3.33}$$

This result is equivalent to Eq. (2.10), as we have from Eq. (3.1)

$$M_{\mu\nu,\text{odd}}=-(3ig^2/4m^2)\delta_{\mu\nu}\,, \tag{3.34}$$

$$M_{\mu\nu,\text{even}}=0 \text{ to this order.} \tag{3.35}$$

We have also computed the next correction to $T_2^{\pm}(p,M)$ which is obtained by doing second Born on Eq. (3.27). The result is, again taking the limit $p\rightarrow 0$ first,

$$\pm\frac{9ig^4}{4\pi^2m^2}+O\left(\frac{1}{M^2}\ln\frac{M}{m}\right), \tag{3.36}$$

where we have used again Eq. (3.20) for $T_1^{\pm}(M)$. From Eqs. (3.1), (3.14), and (3.26) we find the leading term for the forbidden processes to be

$$M_{\mu\nu,\text{even}}=(9ig^4/16\pi^2m^2)\delta_{\mu\nu}\,, \tag{3.37}$$

which gives the quantity α referred to in Eq. (2.11).

We have not shown the convergence of the iteration procedure used here. If the convergence is all right, then we conjecture from Eq. (3.27) that $T^{\pm}$ [Eq. (3.21)] is equal to $\lim_{p\rightarrow 0}T^{\pm}(p)$, where $T^{\pm}(p)=-3ig^2(p^2+m^2)^{-2}\pm 12ig^2(2\pi)^{-4}\int d^4kT^{\pm}(k)k^{-2}[(p-k)^2+m^2]^{-2}$.

We conclude this subsection with a comment on the relation between the present method and the one used previously. $T_1^{\pm}(M)$ is closely related to a quantity which we encountered in I, Sec. VI(d). It was shown there, by the same trace techniques as used here, that the essence of the peratization method lies in the isolation of the nugatory term

$$(-ig^2/m^2)\{1\pm[-(4ig^2/m^2)\Delta_F(0)]\}^{-1} \tag{3.38}$$

in the trace, see I, Eqs. (6.23, 6.24). $\Delta_F(0)$ is the value for zero argument of the Feynman propagator in coordinate space, which is a quadratic divergence. Clearly, the expression just written down corresponds to Eq. (3.30) if in the latter we ignore the g^4 terms. This then establishes a connection between the two methods. In Eq. (3.30) we have, furthermore, an explicit expression for the $O(g^4)$ modifications due to the feedback from low virtual frequencies.

(c) Discussion of the Equation for $\beta^{\pm}(p,M)$

We have now to show that Eq. (3.22) holds true, and therefore return to Eq. (3.20). Unlike the equation for $T^{\pm}(p,M)$, we meet here an inhomogeneous term which goes to zero as $p\rightarrow\infty$. One may therefore ask if Eq. (3.20) can be iterated as it stands. This is impossible, however, to order g^4 one obtains a term $\sim\ln M/m$ in this way. We now show that the β equation can be treated similarly to the T equation by first isolating that term in the kernel which gives rise to the logarithmic singularity just mentioned. This is done as follows. Put

$$\frac{1}{(p-p')^2+m^2}=\frac{1}{p'^2+m^2}[1+y+y^2]+\frac{y^3}{(p-p')^2+m^2},$$
$$y=\frac{2p\cdot p'-p^2}{p'^2+m^2}. \tag{3.39}$$

Perform the angular integrations for the terms which involve y^n, $n\leq 2$ with the help of

$$p^{-4}A_{\mu\nu}(p)\int(p\cdot p')^n(p-p')_\mu(p-p')_\nu\phi(p')d^4p'$$
$$=\frac{3}{4}\int d^4p'\phi(p')\times\begin{cases}1, & n=0\\ -\frac{1}{2}p'^2, & n=1\\ +\frac{1}{4}p'^2(p^2+\frac{1}{3}p'^2), & n=2.\end{cases} \tag{3.40}$$

This yields

$$\frac{A_{\mu\nu}(p)(p-p')_\mu(p-p')_\nu}{p^4[(p-p')^2+m^2]}$$
$$\rightarrow\frac{3}{4}\Bigg\{\frac{p'^4}{3(p'^2+m^2)^3}+\frac{m^2-p^2}{(p'^2+m^2)^2}+\frac{p^4+3p^2p'^2}{(p'^2+m^2)^3}$$
$$\times\frac{4A_{\mu\nu}(p)(p-p')_\mu(p-p')_\nu}{3p^4[(p-p')^2+m^2]}\cdot y^3\Bigg\}. \tag{3.41}$$

The meaning of the arrow in Eq. (3.41) is: the right-hand side of Eq. (3.41) is equivalent to the left-hand side as long as either side is substituted in Eq. (3.20). It is easily checked that it is only the first term on the right-hand side of Eq. (3.41) which gives the ln$M$-singularity when we iterate Eq. (3.20) once.

This leads us to divide β into two parts, as follows:

$$\beta^\pm(p,M)=\beta_1^\pm(p,M)+\beta_2^\pm(p,M) \tag{3.42}$$

$$\beta_1^\pm(p,M)=\pm\frac{4ig^2M^4}{3m^2(2\pi)^4}\int\frac{p'^4}{(p'^2+m^2)^3}\times\frac{[\beta_1^\pm(p',M)+\beta_2^\pm(p',M)]}{(p'^2+M^2)^2} \tag{3.43}$$

$$\beta_2^\pm(p,M)=\frac{-ig^2}{m^2(p^2+m^2)}\pm\frac{16ig^2M^4}{3m^2(2\pi)^4}\int K(p,p')\times\frac{[\beta_1^\pm(p',M)+\beta_2^\pm(p',M)]}{(p'^2+M^2)^2} \tag{3.44}$$

$$K(p,p')=\frac{A_{\mu\nu}(p)(p-p')_\mu(p-p')_\nu}{p^4[(p-p')^2+m^2]}-\frac{1}{4}\frac{p'^4}{(p'^2+m^2)^3}. \tag{3.45}$$

Evidently, $K(p,p')$ is an expression equivalent [with respect to the integral in Eq. (3.20)] to the right-hand side of Eq. (3.41) minus its first term.

The β_1 equation is solved by $\beta_1^\pm(p,M)=\beta_1^\pm(M)$, independent of p. Solving for β_1 in the same manner as described earlier for T_1 we find

$$\beta_1=\frac{c_1g^4\ln(M/m)+c_2g^6\phi_1(M)}{1+c_3g^2M^2\ln(M/m)+c_4g^4\phi_0(M)}, \tag{3.46}$$

where the c's are constants (independent of M). The c_1, c_3 terms are obtained by substituting the g^2 part of β_2 into Eq. (3.43). To this approximation one has $\beta_1\rightarrow 0$ as $1/M^2$. The c_2, c_4 terms come from taking into account the next iteration for β_2. We have

$$\phi_n(M)=M^8\int\frac{p^4d^4p}{(p^2+m^2)^3(p^2+M^2)^2}\times\int\frac{K(p,q)d^4q}{(q^2+M^2)^2(q^2+m^2)^n}. \tag{3.47}$$

The c_2 term is more singular than the c_1 term [cf. a similar situation in Eq. (3.30)], but the c_4 term is also more singular than the c_3 term and the net result is again $\beta\rightarrow 0$ as $1/M^2$.

The kernel $K(p,p')$ is so chosen that we can iterate for β_2. To isolate the terms which survive as $p\rightarrow 0$ it is useful to employ Eq. (3.41). After some straightforward integration one finds

$$\beta^\pm=\lim_{M\to\infty}\lim_{p\to 0}\beta^\pm(p,M)=-\frac{ig^2}{m^4}\pm\frac{ig^4}{8\pi^2m^4}+\cdots. \tag{3.48}$$

Remark. In the derivation of Eq. (3.48) we have had to consider the contribution to $\lim_{p\to 0}\beta_2^\pm(p,M)$ of the β_1 term in Eq. (3.44). This contribution is

$$\sim g^2M^4\beta_1^\pm(M)\int\frac{K(0,p)d^4p}{(p^2+M^2)^2}.$$

With the help of Eq. (3.41) and the M^{-2} behavior of β_1, one finds that this expression is $O(M^{-2})$ as $M\rightarrow\infty$.

To find Eq. (3.22) it would not have been necessary to find $\beta^\pm$ itself. However, this quantity is of interest for other reasons. Let us start from Eq. (2.2) and put in it $p_1=p_2=0$ as well as Eq. (3.12). Bearing in mind the property of $m_{\mu\nu}$ quoted after Eq. (2.2), we see that Eq. (2.2) reduces to Eq. (3.14). Therefore we can find from Eqs. (3.1) and (3.48) the value of β in Eq. (2.2) for the special case of zero energy. For the allowed process we learn nothing new in this way, see Eq. (2.4). But for the forbidden process we now find the result for β quoted in Eq. (2.11). Our present method does not suffice to obtain the energy dependent terms of this form factor.

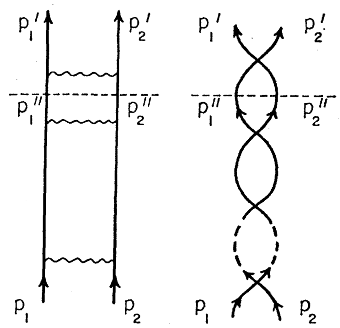

FIG. 1. (a) The general ladder graph in the W theory; (b) the corresponding graphs of the Fermi theory, obtained by pinching the W rungs. The graphs are divided by the dashed horizontal line for the purpose of deriving the BS equations.

4. BS EQUATION FOR FERMI FIELD THEORY

We start from the interaction Lagrangian

$$-(G/\sqrt{2})J_\lambda J_\lambda^*, \qquad (4.1)$$

$$J_\lambda = i\{\bar{\mu}\gamma_\lambda(1+\gamma_5)\nu_\mu + \bar{e}\gamma_\lambda(1+\gamma_5)\nu_e\}.$$

The lowest order matrix element for the "allowed" amplitude is given by (as in I Sec. IV, we omit spinor factors)

$$M^{(1)} = -(G/\sqrt{2})\gamma_\mu^{(1)}\gamma_\mu^{(2)}(1+\gamma_5^{(1)})(1+\gamma_5^{(2)}). \qquad (4.2)$$

Just as for the W theory, one can distinguish between allowed and forbidden processes; the latter have no contributions of order lower than G^2. We consider now those graphs in the Fermi theory which are the analogs of the uncrossed ladder graphs in the W theory. The correspondence between the graphs is as drawn in Fig. 1.

As before, one defines amplitudes M_{odd} and M_{even} for the allowed and forbidden processes and again introduces the linear combinations $M_\pm$ as in Eq. (3.1). Transcribing step by step the arguments of I Sec. IV, one obtains the BS equation

$$M_\pm(p_1',p_2',p_1,p_2) = M^{(1)} \pm \frac{G}{\sqrt{2}}\frac{1}{(2\pi)^4}\int d^4p_1' \times\gamma_\mu^{(1)}(1+\gamma_5^{(1)})\gamma_\mu^{(2)}(1+\gamma_5^{(2)}) \times\frac{1}{(p_1'')_R^2}\frac{1}{(p_2'')_R^2}M_\pm(p_1'',p_2'',p_1,p_2). \qquad (4.3)$$

The regulated fermion propagators have been defined in Eq. (3.11), $M^{(1)}$ is given by Eq. (4.2).

Put

$$M_\pm = N_\pm\gamma_\mu^{(1)}(1+\gamma_5^{(1)})\gamma_\mu^{(2)}(1+\gamma_5^{(2)}). \qquad (4.4)$$

Then[10]

$$\gamma_\mu^{(1)}\gamma_\mu^{(2)}N_\pm(p_1',p_2',p_1,p_2) = -\gamma_\mu^{(1)}\gamma_\mu^{(2)}(G/\sqrt{2}) \pm\frac{4G}{\sqrt{2}}\xi_{\mu\lambda\rho\tau}\xi_{\mu\eta\rho\sigma}\frac{1}{(2\pi)^4}\int d^4p''\, p_{1\lambda}''p_{2\eta}''\gamma_\tau^{(1)}\gamma_\sigma^{(2)} \times\frac{1}{(p_1'')_R^2}\frac{1}{(p_2'')_R^2}N_\pm(p_1'',p_2'',p_1,p_2). \qquad (4.5)$$

[10] We have used Eq. (IA.4) at this point.

With the help of Eq. (3.8), it follows that

$$N_\pm(p_1',p_2',p_1,p_2) = -\frac{G}{\sqrt{2}} \pm \frac{16G}{\sqrt{2}}\frac{1}{(2\pi)^4}\int d^4p_1'' \times\frac{(p_1''\cdot p_2'')}{(p_1'')_R^2(p_2'')_R^2}N_\pm(p_1'',p_2'',p_1,p_2). \qquad (4.6)$$

Because of Eq. (3.4) we see that the only dependence of the kernel on the external momenta is via the total energy momentum vector $p_1+p_2\equiv P$. Thus $N_\pm$ may be taken to depend on P only. But this makes it a constant with respect to the p_1'' integration. Thus the solution is

$$N_\pm(P) = -\frac{G}{\sqrt{2}}\left(1\mp\frac{16GM^4}{\sqrt{2}(2\pi)^4} \times\int\frac{d^4k\{k(P-k)\}}{k^2(k-P)^2(k^2+M^2)\{(k-P)^2+M^2\}}\right)^{-1}. \qquad (4.7)$$

Evidently the k integral diverges like M^2 for $M\to\infty$, so that

$$N_\pm(P) = 0 \quad \text{for all } P, \qquad (4.8)$$

which leads to the result quoted in Eq. (2.12). Note that equations like (4.7) are closely related to earlier results by Abrikosov *et al.*[7] on Fermi field theories.

While there is therefore no need to consider separately the specific case $p_1=p_2=0$, it is nevertheless instructive for a comparison with the W theory to make this specialization in Eq. (4.6). Using again the notations given in Eq. (3.12), Eq. (4.6) reduces to

$$N^\pm(p) = \frac{-G}{\sqrt{2}} \pm \frac{16GM^4}{\sqrt{2}}\int\frac{d^4p'\cdot N^\pm(p')}{p'^2(p'^2+M^2)^2}. \qquad (4.9)$$

This equation is very much akin to the one given in Eq. (3.26) for the high-frequency part of the trace. That is, there is a correspondence between $N^\pm$ and $T_1^\pm$. And the solution: $T_1^\pm=$ constant which $\to 0$ for $M\to\infty$ corresponds precisely to the solution (4.8) of the Fermi theory. Indeed, if we make the curious limiting process: $m\to\infty$ *before* performing virtual momentum integrations, then Eq. (3.17) becomes *identical* with Eq. (4.9) if use is made of $g^2m^{-2}\to 2^{-1/2}G$.

Whatever the results of this section may mean, it is hoped that they have helped to make it abundantly clear that the Fermi-theory and the W-theory have a quite distinct outlook inasfar as the peratization program is concerned.

ACKNOWLEDGMENT

We would like to thank Dr. T. T. Wu for stimulating discussions.

APPENDIX A

We list here a few elementary Feynman integrals which play a role in the foregoing sections. Put

$$\Phi_j(M,m)=\int\frac{d^4p}{p^2(p^2+M^2)^2(p^2+m^2)^j}.$$

We have used

$$\Phi_0(M,m)=i\pi^2/M^2\,,$$

$$\Phi_1(M,m)=\frac{2i\pi^2}{M^2-m^2}\left[\frac{1}{M^2-m^2}\ln\frac{M}{m}-\frac{1}{2M^2}\right],$$

$$\Phi_2(M,m)=\frac{2i\pi^2}{(M^2-m^2)^2}\left[-\frac{2}{M^2-m^2}\ln\frac{M}{m}+\frac{1}{2M^2m}\right].$$

in the discussion of $T^{\pm}$.

APPENDIX B: MOMENTUM DEPENDENT CORRECTIONS TO THE MATRIX ELEMENT

In the previous sections, we have shown that the sum of the uncrossed ladders for lepton scattering at zero external momenta can be obtained from the integral equation (3.15), and agrees with the result obtained in I. The question naturally arises as to whether the matrix element at finite momentum transfer can also be obtained. In particular, we would like to know whether the momentum dependent corrections are finite, and if so what is their order in g^2. We shall restrict our selves to external momenta which are still "small" in the sense of Eq. (I6.11), i.e., $g^2(p^{\text{ext}})^2m^{-2}<1$. In this region, it is reasonable to expand the matrix element in powers of $(p^{\text{ext}})^2m^{-2}$, and to consider the behavior of the separate coefficients.

To do this, we shall need a slight generalization of the power counting argument of I, Sec. 3. In order to give this generalization, it is necessary to be able to determine the degree of divergence of a Feynman graph, such as an uncrossed ladder graph. It would of course be easy to estimate this if the answer were found by subtracting the total number of momenta in the denominator of the integral from the number in the numerator. We shall call this "over-all power counting." However, it is clear that this procedure is not valid in general for multiple integrals of the type of an nth order graph. In particular, there arises the problem of divergent subintegrations, which can change the degree of divergence of the graph as a whole. This problem also arises in the renormalization theory of quantum electrodynamics,[11] where it is treated through the concept of primitively divergent graphs.[11] However, that idea does not appear to go through here, and in particular the uncrossed ladder graphs, which are irreducible in the sense of renormalization theory, are not primitively divergent, and hence we cannot conclude that all subintegrations are finite. Nevertheless, the method of over-all power counting gives the right answer when we confine ourselves to the leading singular terms of the matrix elements for the case of zero external momentum. This was the case considered in I, Sec. III, where we found that for an $(n+1)$-rung graph, $(n\geqslant 1)$, the leading term is $\sim g^{2n+2}(M/m)^{2n}$. That this result is indeed correct can be seen by a formal expansion of the denominator in Eq. (3.30).

However, already for the next-to-leading singularity for zero external momentum, over-all power counting breaks down. Indeed, this method would lead one to anticipate next to leading contributions $\sim g^{2n+2}(M/m)^{2n-2}$ (for zero-lepton mass), the argument being that the next to leading term has two more powers of m, hence two powers less in M. This would correspond to a $\ln M$ singularity to $O(g^4)$ and power singularities for $n\geq 1$. However, the formal expansion of Eq. (3.30) shows that actually the next-to-leading singularities are $\sim g^{2n+2}(M/m)^{2n-2}\ln(M/m)$ *for all n*.

Let us now turn to the momentum-dependent terms $\sim(p^{\text{ext}}/m)^2$. *If* over-all power counting were applicable to these terms, their leading singular contributions would be $\sim g^{2n+2}(M/m)^{2n-2}(p^{\text{ext}}/m)^2$, $n\geqslant 1$. In what follows we explore the consequences of the assumption that this estimate is correct. (To check whether this is true or not, one needs a more detailed study of higher order graphs than we have made here.) These consequences are twofold.

(1) For the allowed processes, at small momenta, the momentum-dependent terms are small in comparison with the terms we have computed.

(2) This is not the case for the forbidden processes, however, where the coefficient of the constant term is actually of higher order in g than the $(p^{\text{ext}})^2$ term. The above estimate leads in fact to contributions $\sim g^4(\ln g)(p^{\text{ext}}/m)^2$ where the term in question can be calculated from the one graph which involves two uncrossed meson exchange.[12]

The Feynman matrix element for this graph is given by the following expression, obtained by regularizing the W propagator as in I. We follow our previous notations for the external momenta.

$$M_4=\frac{g^4}{(2\pi)^4}\int\bar{u}_1'\gamma_\mu(1+\gamma_5)\frac{1}{p_1-k}\gamma_\nu(1+\gamma_5)u_1\bar{u}_2'\gamma_\tau(1+\gamma_5)\frac{1}{p_2+k}\gamma_\lambda(1+\gamma_5)u_2d^4k$$
$$\times\left[\delta_{\mu\tau}+\frac{(q-k)_\mu(q-k)_\tau}{m^2}\right]\left(\delta_{\nu\lambda}+\frac{k_\nu k_\lambda}{m^2}\right)\left\{\frac{(M^2-m^2)^2}{(k^2+m^2)(k^2+M^2)[(k-q)^2+m^2][(k-q)^2+M^2]}\right\}.\quad\text{(B1)}$$

[11] F. J. Dyson, Phys. Rev. **75**, 1736 (1949).

[12] An argument of this kind was used by T. D. Lee, Phys. Rev. **128**, 899 (1962), to compute the $\alpha\ln\alpha$ corrections in W electrodynamics.

We are interested in the terms proportional to two powers of external momenta, and in particular to the logarithmically divergent part of this. It is easy to see that such terms come only from the term proportional to $(q-k)_\mu(q-k)_\tau k_\nu k_\lambda$. We therefore select this term, and rewrite it as

$$\frac{4g^4}{(2\pi)^4m^4}\int \bar{u}_1'(\boldsymbol{p}_1-\boldsymbol{p}_1'-\boldsymbol{k})\frac{1}{\boldsymbol{p}_1-\boldsymbol{k}}\boldsymbol{k}(1+\gamma_5)u_1\bar{u}_2'(\boldsymbol{p}_2'-\boldsymbol{p}_2-\boldsymbol{k}) \times\frac{1}{\boldsymbol{p}_2+\boldsymbol{k}}\boldsymbol{k}(1+\gamma_5)u_2\frac{(M^2-m^2)^2}{(k^2+m^2)(k^2+M^2)[(q-k)^2+m^2][(q-k)^2+M^2]}. \quad \text{(B2)}$$

For simplicity, we calculate this on the mass shell in the approximation where the masses of the external fermions are also neglected. [We have already neglected the masses of the virtual fermions in writing (B1)]. In this case (B2) simplifies tremendously, and becomes

$$[4g^4/(2\pi)^4m^4]\bar{u}_1'\gamma_\mu(1+\gamma_5)u_1\bar{u}_2'\gamma_\nu(1+\gamma_5)u_2\int\frac{k_\mu k_\nu(M^2-m^2)^2d^4k}{(k^2+m^2)(k^2+M^2)[(q-k)^2+m^2][(q-k)^2+M^2]}. \quad \text{(B3)}$$

We notice that in this approximation, the integral only depends on the momentum transfer q. Furthermore, the term $\beta q_\mu q_\nu$ in Eq. (2.2) gives zero in this approximation. Therefore, the expression (B3) contributes only to $\alpha(q^2)$ of Eq. (2.2).

The integral can be evaluated by the standard technique of introducing Feynman parameters. Upon doing this, and expanding the resultant integral in powers of q^2, we obtain the following results. There is a constant term proportional to M^2, which has been included in our zero energy solution (3.37). There is also a term proportional to q^2 with a coefficient proportional to $\ln(M/m)$. Specifically, this term is

$$\frac{-ig^4q^2}{24\pi^2m^4}\ln\left|\frac{M}{m}\right|\bar{u}_1'\gamma_\mu(1+\gamma_5)u_1\bar{u}_2'\gamma_\mu(1+\gamma_5)u_2. \quad \text{(B4)}$$

The remaining terms are finite. We see that there results a term, in the forbidden amplitude, which is proportional to q^2/m^2, and of order $g^2\ln(1/g)$. The contribution of this term to the quantity $M_{\mu\nu,\mathrm{even}}$ is

$$\frac{-g^4i}{24\pi^2m^2}\frac{q^2}{m^2}\ln\frac{1}{g}\delta_{\mu\nu}. \quad \text{(B5)}$$

This expression agrees with the corresponding term in I, Eq. (6.16), calculated by solving the approximate integral equation.

The results of the text and this Appendix may be combined to give expressions for the allowed and forbidden amplitudes, neglecting the fermion masses.

$$M_{\mu\nu,\mathrm{odd}}=(-3ig^2/4m^2)\delta_{\mu\nu}+O(g^4), \quad \text{(B6)}$$

$$M_{\mu\nu,\mathrm{even}}=(9ig^4/16\pi^2m^2)\delta_{\mu\nu} +(g^4i\ln g/24\pi^2m^4)q^2\delta_{\mu\nu}+O(g^4)P_{\mu\nu}, \quad \text{(B7)}$$

where the tensor $P_\mu{}^\nu$ vanishes at zero momentum.